Introducción al análisis semiclásico

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Plan del cursillo

1. Teoría Espectral de Operadores y Mecánica Cuántica
2. Cuantización y Operadores Pseudodiferenciales
3. Cálculo funcional para operadores pseudodiferenciales
4. Estudio de valores propios

Plan de las sesiones I y II

1. Operadores lineales en espacios de Hilbert
2. Teorema Espectral (Cálculo funcional para operadores autoadjuntos)
3. Noción de observable cuántico, estado y ecuación de Schrödinger

Operadores lineales en espacios de Hilbert (1)

Definición 1

Un espacio de Hilbert $\mathcal{H}$ *es un espacio vectorial complejo, con un producto interno* $\langle \cdot, \cdot \rangle$ *que es completo para la norma inducida por el producto interno* $||f|| = \langle f, f \rangle^{\frac{1}{2}}$*, y separable.*

Comentario 1

Separable en este contexto es equivalente a tener una base numerable. Es decir que con esta definición todos los Espacios de Hilbert de dimensión infinita son isomorfos.

Ejemplo 1

- $\mathbb{C}^n$ *con* $\langle \alpha, \beta \rangle = \sum_{j=1}^{n} \overline{\alpha_j} \beta_j$
- $l^2(\mathbb{Z})$ *con* $\langle \alpha, \beta \rangle = \sum_{n \in \mathbb{Z}} \overline{\alpha(n)} \beta(n)$
- $L^2(\mathbb{R}^n)$ *con* $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x)$

Operadores lineales en espacios de Hilbert (2)

Definición 2

Una aplicación lineal $B : \mathcal{H} \to \mathcal{H}$ *se dice acotada (o continua) si existe* $C > 0$ *tal que*

$$||Bf|| \leq C||f|| \tag{1}$$

para todo $f \in \mathcal{H}$. *Notamos por* $\mathcal{B}(\mathcal{H})$ *el conjunto de operadores acotados en el espacio de Hilbert* $\mathcal{H}$. *Además definimos la norma,*

$$||B|| = \sup_{f \in \mathcal{H}} \frac{||Bf||}{||f||} \tag{2}$$

Ademas, para todo $B \in \mathcal{B}(\mathcal{H})$ *existe un único* $B^* \in \mathcal{B}(\mathcal{H})$ *que satisface*

$$\langle B^* f, g \rangle = \langle f, Bg \rangle \ . \tag{3}$$

Operadores lineales en espacios de Hilbert (3)

Decimos que B^* es el adjunto de B. Se tiene ademas las siguientes propiedades.

$$||B^*|| = ||B|| \quad ; \quad (B^*)^* = B \quad ; \quad (AB)^* = B^*A^* \quad ; \quad ||B^*B|| = ||B||^2 \,.$$

Comentario 2

Respecto de la composición de operadores, $\mathcal{B}(\mathcal{H})$ es un álgebra. Como es completa respecto de su norma es una álgebra de Banach. Como ademas la involución satisface $||B^*B|| = ||B||^2$ es una C^*-algebra.

Proposición 1

Todo operador acotado B se puede escribir como $B = U|B|$ donde $|B|$ es un operador positivo y U es un isometria parcial (isometria fuera de su kernel). Un operador positivo cumple que $\langle f, Bf \rangle \geq 0$ y en particular es auto adjunto.

Algunos tipos de operadores

- B es autoadjunto si $B^* = B$
- B es normal si $B^*B = BB^*$
- B es una projección ortogonal si $B^2 = B = B^*$
- B es unitario si $BB^* = B^*B = 1$

Consideremos $\mathcal{H} = l^2(\mathbb{Z})$. Si $f : \mathbb{Z} \to \mathbb{R}$ es acotada como función, entonces el operador de multiplicación por f

$$(M_f\alpha)(n) = f(n)\alpha(n) \tag{4}$$

es un operador acotado y autoadjunto. Dado cualquier vector $g \in l^2(\mathbb{Z})$, la projección ortogonal en g es

$$|g\rangle\langle g|\alpha = \langle g, \alpha\rangle g \ . \tag{5}$$

El operador S definido por

$$(S\alpha)(n) = \alpha(n+1) \tag{6}$$

es unitario. En efecto su adjunto cumple

$$(S^*\alpha)(n) = \alpha(n-1) \ . \tag{7}$$

Operadores Compactos

Definición 3

Si el rango de un operador A es de dimensión finita decimos que es un operador de rango finito y notamos el conjunto de estos operador por $\mathcal{F}(\mathcal{H})$. Si $A \in \mathcal{F}(\mathcal{H})$, entonces existen $\{g_j, h_j\}_{j=1}^{N} \subset \mathcal{H}$ tal que

$$A = \sum_{j=1}^{N} |h_j\rangle\langle g_j| \ . \tag{8}$$

Definición 4

Decimos que $B \in \mathcal{B}(\mathcal{H})$ es un operador compacto si es limite en norma de una sucesión de operadores de rango finito. Notamos el conjunto de los operadores compactos por $\mathcal{K}(\mathcal{H})$.

Propiedades de los operadores compactos

- $B \in \mathcal{K}(\mathcal{H})$ implica $B^* \in \mathcal{K}(\mathcal{H})$
- $A \in \mathcal{B}(\mathcal{H})$ y $B \in \mathcal{K}(\mathcal{H})$ implica $AB, BA \in \mathcal{K}(\mathcal{H})$
- Se tiene entonces que $\mathcal{K}(\mathcal{H})$ es un ideal bilatero de $\mathcal{B}(\mathcal{H})$

Definición 5

Un operador compacto se dice de Hilbert–Schmidt si

$$||A||_{\mathcal{B}_2} := \sum_{j \in \mathbb{N}} ||Ae_j||^2 < \infty \tag{9}$$

para una (cualquier) base ortonormal $\{e_j\}_{j \in \mathbb{N}}$. *Notamos por* $\mathcal{B}_2$ *la clase de operadores de Hilbert-Schimidt.*

Se cumple que para todo $B \in \mathcal{B}(\mathcal{H})$ y $A \in \mathcal{B}_2$

$$||AB||_{\mathcal{B}_2} \leq ||B|| \, ||A||_{\mathcal{B}_2} \tag{10}$$

Operadores de clase traza

Para todo $B \geq 0$ existe un único $B^{\frac{1}{2}}$ tal que $B^{\frac{1}{2}} B^{\frac{1}{2}} = B$.

Definición 6

Un operador se dice de clase traza si

$$||A||_{\mathcal{B}_1} := \sum_{j \in \mathbb{N}} \left|\left| \, |A|^{\frac{1}{2}} e_j \right|\right|^2 = \sum_{j \in \mathbb{N}} \langle e_j, |A| e_j \rangle < \infty \ . \tag{11}$$

Para todo $A \in \mathcal{B}_1$ definimos

$$\operatorname{Tr}(A) = \sum_{j \in \mathbb{N}} \langle e_j, A e_j \rangle \ . \tag{12}$$

En resumen tenemos:

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{B}_j \subset \mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \ . \tag{13}$$

Operadores (no acotados)

Definición 7

Un operador en un espacio de Hilbert $\mathcal{H}$ es un par $(A, D(A))$, con $D(A)$ un subespacio denso de $\mathcal{H}$ y $A : D(A) \to \mathcal{H}$ una aplicación lineal.

Por ejemplo, en $L^2((0,1))$ el "operador" $-i\frac{\partial^2}{\partial x^2}$ tiene diferentes dominios de interés:

- $C_c^\infty((0,1))$ el espacio de funciones infinitamente diferenciables con soporte compacto
- El espacio de funciones infinitamente diferenciables tales que su derivadas se anulan en 0 y 1.

Conjunto resolvente y espectro de un operador

Definición 8

Un operador $(A, D(A))$ *es cerrrado si para toda sucesión convergente de vectores* $\{f_n\}$ *y tal que* $\{Af_n\}$ *sea una sucesion de Cauchy, se tiene que* $\lim_{n\to\infty} f_n = f_\infty \in D(A)$ *y* $\lim_{n\to\infty} Af_n = Af_\infty$

Definición 9

Sea $(A, D(A))$ *un operador cerrado. El conjunto resolvente* $\rho(A)$ *se define por*

$$\rho(A) = \{z \in \mathbb{C} | (A - z)^{-1} \in \mathcal{B}(\mathcal{H})\} \ . \tag{14}$$

El operador acotado $(A - z)^{-1} : \mathcal{H} \to D(A)$ *se dice la resolvente de* A *en* z*. Se puede demostrar que* $\rho(A)$ *es un conjunto abierto (de* $\mathbb{C}$*). Se sigue que el espectro de* A*, definido por*

$$\sigma(A) := \mathbb{C} \backslash \rho(A) \tag{15}$$

es cerrado.

Adjunto (versión no acotada)

Dado un operador $(A, D(A))$ su adjunto $(A^*, D(A^*)$ se define por

$$D(A^*) = \{f \in \mathcal{H} | \exists f^* \in \mathcal{H} \text{ tal que} \langle f, Ag \rangle = \langle f^*, g \rangle \forall g \in D(A)\} \quad (16)$$

y $A^* f = f^*$.

Lema 1

$((A^*), D(A^*))$ *es un operador cerrado y* $\mathrm{Ker}(A^*) = (\mathrm{Ran}(A))^{\perp}$.

Definición 10

$(A, D(A))$ *es autoadjunto si* $(A, D(A)) = (A^*, D(A^*))$.

Teorema 1

Si $(A, D(A))$ *es autoadjunto entonces* $\sigma(A) \subset R$

Medida espectral

Definición 11

Una familia espectral es una familia de proyecciones ortogonales $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ que satisface

- $E_\lambda E_\mu = E_{\text{mín}\{\lambda,\mu\}}$
- *Continua por la derecha*
- $\text{lím}_{\lambda\to-\infty} E_\lambda =$ *y* $\text{lím}_{\lambda\to\infty} E_\lambda = 1$

Una familia espectral define una medida espectral E, esto es una medida a valores proyecciones mediante $E((a,b]) := E_b - E_a$.

Teorema 2

Dado un operador $(A, D(A))$ autoadjunto. A toda función φ continua y acotada sobre $\sigma(A)$ le corresponde un operador $\varphi(A)$ definido por

$$\varphi(A) = \int_{\sigma(A)} \varphi(\lambda) E(d\lambda) \ . \tag{17}$$

Además, si consideramos la estrucutra de C^*-algebra en $C_b(\sigma(A))$ (multiplicación puntual, conjugación compleja y norma del supremo) tenemos que $\varphi \to \varphi(A)$ es un morfismo de C^*-algebras y en particular es contractivo. Es decir:

$$||\varphi(A)|| \leq ||\varphi||_\infty \ . \tag{18}$$

Tipos de espectro

A partir de la medida espectral, y para cada $f \in \mathcal{H}$, definimos una medida por $m_f(O) = ||E(O)f||^2$. De aqui podemos deducir una descomposición de $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_p \ .$$

tal que si $f \in \mathcal{H}_{ac}$, la medida m_f es absolutamente continua con respecto a la medida de Lebesgue. Analogamente se define la parte singular continua $\mathcal{H}_{sc}$ y la parte puntual $\mathcal{H}_p$. En particular, si f es un vector propio, entonces $f \in \mathcal{H}_p$.

Observables cuánticos

Un sistema cuantico está descrito por un vector (unitario) de un espacio de Hilbert Un observable cuántico es un operador autoadjunto A. El espectro $\sigma(A)$ son los valores posibles que puede tomar el observable A. Si E es la medida espectral asociada a A, φ es el estado del sistema y O es un conjunto medible, la probabilidad de que el observable A exhiba un valor en O es

$$||(\int_O E(d\lambda))\psi||^2 \ . \tag{19}$$

Ejemplo 2

En $L^2(\mathbb{R})$ *los dos principales observables que necesitamos son el operador posición* $(Q\varphi)(x) = x\varphi(x)$ *y el observable momentum* $(-i\partial_x\varphi)(x) = -i\varphi'(x)$*. Son la cuantización los observables clásicos posición y momentum.*

Relación entre observables clásicos y cuánticos

El movimiento de una particula clásica esta descrita por un curva $t \to (x(t), \xi(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$. La energía total está dada

$$E = \frac{1}{2m}\xi(t)^2 + V(x(t)) \ . \tag{20}$$

Pasando al lado cuántico, E corresponde al operador de Schrödinger

$$H = -\frac{h}{2m}\Delta + V(x) \ . \tag{21}$$

Un observable clásico es una función infinitamente diferenciable sobre el espacio de fase $\mathbb{R}^3 \times \mathbb{R}^3$. Dado un observable clásico $a(x, \xi)$, ¿cómo le podemos asignar un observable cuántico?

Operadores pseudodiferenciales

Sea a un simbolo clásico. Formalmente definimos $a^w(x, hD) = \mathrm{Op}_h^w(a)$ por la siguiente formula donde u es un vector adecuado de $L^2(\mathbb{R}^3)$

$$[\mathrm{Op}_h^w(a)u](x) = \frac{1}{(2\pi h)^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) dy\, d\xi \ . \tag{22}$$

Ejemplo 3

Si $a(x, \xi) = \sum_{|\alpha| \le m} b_\alpha(x)\xi^\alpha$ *entonces*

$$\mathrm{Op}_h^w\Big(\sum_{|\alpha|\le m} b_\alpha(x)\xi^\alpha\Big) = \sum_{|\alpha|\le m} (hD)^\alpha b_\alpha(x) \tag{23}$$

es un operador diferencial de orden m.

Espacio de simbolos

Dado $x \in R^n$ definimos el corchete japonés por $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$.

Definición 12

Decimos que $m : \mathbb{R}^n \to \mathbb{R}_+$ *es una función de orden si existen constantes* $C_0 > 0$ *y* N_0 *tales que* $m(x) \leq \langle x - y \rangle_0^N m(y)$

Definición 13

$S(\mathbb{R}^n, m) = S(m)$ *es el espacio de funciones* $a \in C^\infty(\mathbb{R}^n)$ *talque para todo* $\alpha \in \mathbb{N}^n$, *existe* $C_\alpha > 0$ *talque*

$$|\partial^\alpha a(x)| \leq C_\alpha m(x) \ . \tag{24}$$

Decimos que $\sum_j a_j h^j$ converge en el límite semiclásico a un simbolo a si $a - \sum_{j=1}^{N}$ es de la forma $h^{N+1} S(m)$ para todo N.

Introducción al análisis semiclásico

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Lican Ray
May 28-31, 2024

Plan

1. **Introduction : classical mechanics, quantum mechanics and semiclassical mechanics**
2. **The functional calculus of h-pseudodifferential operators via the Helffer-Sjöstrand formula**
3. **Trace formula for h-pseudodifferential operators**
4. **Semiclassical Weyl's law**
5. **Potential wells in 1D**

The initial goal of semiclassical analysis is to explore a central problem in physics which is the study of the relationship between classical mechanics and quantum mechanics. The starting point is the following famous principle :

> **Bohr's correspondence principle (1923) :** One should recover the classical mechanics from the quantum mechanics when the Planck constant $\hbar$ becomes negligible. In other words, classical mechanics is the limit of quantum mechanics when Planck's constant tends to zero.

Classical mechanics

We start from a C^∞ function on $\mathbb{R}^{2d}$: $(x,\xi) \mapsto p(x,\xi)$ which will permit to describe the motion of the system in consideration and is called the classical Hamiltonian. The variable x corresponds in the simplest case to the position and ξ to the impulsion of one particle. The evolution is then described, starting from a given point (y,η), by the so called Hamiltonian equations

$$\frac{d}{dt}x(t) = \partial_\xi p(x(t),\xi(t)), \quad \frac{d}{dt}\xi(t) = -\partial_x p(x(t),\xi(t)), \quad (x(t),\xi(t))_{|t=0} = (y,\eta) \in \mathbb{R}^{2d}.$$

The classical trajectories are then defined as the integral curves of a vector field defined on $\mathbb{R}^{2d}$ called the hamiltonian vector field associated with p and defined by $H_p = (\partial_\xi p, -\partial_x p)$:

$$\exp(tH_p)(y,\eta) = (x(t),\xi(t)) : \text{ hamiltonian flow} \quad (t \in \mathbb{R}).$$

The time evolution of a classical observable $q \in C^\infty(\mathbb{R}^{2d};\mathbb{R})$ given by $q_0(t,x,\xi) = q(exp(tH_p)(x,\xi))$, $t \in \mathbb{R}$, $(x,\xi) \in \mathbb{R}^{2d}$, is described by the equation

$$\frac{d}{dt}q_0(t,x,\xi) = \{p,q\}\left(exp(tH_p)(x,\xi)\right), \quad q_0(t,x,\xi)_{|t=0} = q(x,\xi) \tag{1}$$

where $\{p,q\} := \partial_\xi p \cdot \partial_x p - \partial_x p \cdot \partial_\xi q$ is the Poisson bracket of p, q.

In the framework of the classical mechanics the main questions could be :

- Are the trajectories bounded ?
- Are there periodic trajectories ?
- Is the energy surface compact ?

The solutions of the above questions could be very difficult. Let us just recall the conservation of energy law

$$p(\exp(tH_p)(x,\xi)) = p(x,\xi).$$

This means that for any energy $E \in \mathbb{R}$, the energy surface $p^{-1}(E) = \{(x,\xi) \in \mathbb{R}^{2d}; p(x,\xi) = E\}$ is stable by the hamiltonian flow $\exp(tH_p)$.

Quantum mechanics

The quantum theory is born around 1920. In quantum mechanics, our basic object will be a (possibly non-bounded) self-adjoint operator defined on a dense subspace of an Hilbert space $\mathcal{H}$. In order to simplify we shall always take $\mathcal{H} = L^2(\mathbb{R}^d)$.
This operator (quantum hamiltonian) can be associated with a symbol (classical hamiltonian) p by different techniques called quantizations. For instance one can work with the Weyl quantization :

$$C^\infty(\mathbb{R}^{2d};\mathbb{R}) \ni p \longmapsto \mathrm{Op}_h^w(p) \text{ a self-adjoint operator in } L^2(\mathbb{R}^d).$$

The operator $P = \mathrm{Op}_h^w(p)$ is called h-pseudodifferential operator of symbol p. Here $h > 0$ is a small parameter which plays the role of the Planck constant called the semiclassical parameter.

Given such an operator, the dynamics of the quantum system is governed by the Schrödinger equation

$$ih\frac{d}{dt}\psi(t) = P\psi(t), \quad \psi(t)_{t=0} = \psi_0 \in L^2(\mathbb{R}^d).$$

This equation generates a unitary operator called the evolution operator (or the propagator) $e^{-\frac{it}{h}P}$:

$$\psi(t) = e^{-\frac{it}{h}P}\psi_0 \quad (t \in \mathbb{R}).$$

The evolution in time of a quantum observable $\mathrm{Op}_h^w(q)$ given by

$$Q(t) := e^{\frac{it}{h}P}\mathrm{Op}_h^w(q)e^{-\frac{it}{h}P}$$

is then described by the quantum analogous of equation (1) the so called Heisenberg equation

$$\frac{d}{dt}Q(t) = \frac{i}{h}[P, Q(t)], \quad Q(t)_{|t=0} = \mathrm{Op}_h^w(q).$$

Semiclassical mechanics

A rigorous mathematical justification of the correspondence principle is given by the following theorem known as Egorov's theorem :

$$Q(t) = \mathrm{Op}_h^w(q(\exp(tH_p))) + \mathcal{O}_{\mathcal{L}(L^2)}(h)$$

uniformly for $|t| \leq T$, for all $T \in \mathbb{R}$. This result of course needs appropriate assumptions on the symbols p and q.

Semiclassical analysis is thus an asymptotic analysis which allows to establish the relation between quantum objects and classical objects in the the semiclassical limit $h \to 0^+$:

Quantum objects	Corresp. in the semiclassical limit	Classical objects
Quantum objects : quantum Hamiltonians (operators), eigenvalues, eigenfunctions, quantum resonances, resonance states, etc	**Corresp. in the** $\longrightarrow$ **semiclassical limit**	**Classical objects** : classical hamiltonians (functions), volumes in the phase space, closed trajectories of the hamiltonian flow, Maslov's index, etc

To establish this correspondance, we often use techniques from microlocal analysis, pseudodifferential calculus, Fourier Integral operators, symplectic geometry, etc. The techniques of semiclassical analysis have found many applications in several fields where a small parameter appears naturally and plays an important role, for instance :

- The square root of the quotient between the electronic and nuclear masses in the Born-Oppenheimer approximation in molecular dynamics.
- The adiabatic parameter in adiabatic theory.
- The inverse of the square root of the position in spectral problems in high-energy regime.
- The inverse of the norm of the position in scattering theory.
- The inverse of the temperature in the study of the metastability (Langevin equation).

A prototype result in semiclassical analysis has the following form : Given a pseudodifferential operator $\mathrm{Op}_h^w(p)$ with a symbol p defined on $\mathbb{R}^{2d}$ and denoting H_p the corresponding hamiltonian field, under some reasonable assumptions on the symbol p, we have, in the semiclassical limit $h \to 0^+$,

A geometric property of the hamiltonian field H_p	$\Rightarrow$	A spectral property of the operator $\mathrm{Op}_h^w(p)$.

In order to study the dynamical and spectral properties of h-pseudodifferential operators one needs to establish a functional calculus on these operators. What is functional calculus ? Any formula which represents a function as a superposition of simpler functions can (at least in principle) be taken as a starting point for a functional calculus.

Recall that for a self-adjoint operator P on a Hilbert space $\mathscr{H}$ and a "nice" real-valued function f, say a bounded continuous function, the spectral theorem allows us to define a new linear bounded self-adjoint operator $f(P)$ as follows

$$f(P) = \int_{\mathbb{R}} f(t) dE_t$$

where $E_t = 1_{(-\infty,t]}(P)$ is the family of spectral projections associated with P.

This formula is very useful in many problems in spectral theory. However, it is unfortunately a bit too abstract to work with ! In particular, in the framework of semiclassical pseudodifferential operators, it is not convenient to answer the following natural questions :

- Is $f(P)$ a semiclassical pseudodifferential operator if P is a semiclassical pseudodifferential operator and f is a "nice" function ?
- If yes, do we have an algorithm to compute the symbol of $f(P)$ from that of P ?
- If "yes and yes", is this formula good enough to use it in the study of the spectral properties of the operator P ?

In the following we will introduce a formula for $f(P)$ called the Helffer-Sjöstrand formula which allows us to answer the above questions. As we will see this formula is very useful in the spectral analysis of h-pseudodifferential operators since it relates $f(P)$ with the resolvent $(z - P)^{-1}$.

Almost analytic extensions

In all the talk $C_0^\infty(\mathbb{R})$ stands for the space of C^∞ real-valued functions on $\mathbb{R}$ with compact support.

For a function defined on $\mathbb{C}$ (or $\mathbb{R}^2$) we will use the operator

$$\overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

We recall that if a function g is holomorphic on some open subset $\Omega \subset \mathbb{C}$, then $\overline{\partial} g \equiv 0$ in Ω.

Definition. Let $f \in C_0^\infty(\mathbb{R})$. We call almost analytic extension of f any function $\widetilde{f} \in C_0^\infty(\mathbb{C})$ satisfying

1) $\widetilde{f}_{|\mathbb{R}} = f$.
2) For all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|\overline{\partial}\widetilde{f}(z)| \leq C_N |\mathrm{Im}\, z|^N.$$

- The first condition just states that the restriction of $\widetilde{f}$ on the real line coincides with f.
- The second condition is equivalent to the fact that $\overline{\partial}\widetilde{f}$ vanishes at infinite order on the real line, that is, if one identifies $\mathbb{C} \ni z = x + iy \mapsto (x, y) \in \mathbb{R}^2$:

$$\partial_y^k \overline{\partial}\widetilde{f}(x, 0) = 0, \quad \forall k \in \mathbb{N}, x \in \mathbb{R}.$$

The notion of almost analytic extension was introduced by **Hörmander** in the 60's of the previous century and has subsequently been used by many people : **Nirenberg**, **Maslov**, **Kucherencko**, **Dynkin**, **Taylor**, etc.

Almost analytic extensions

Given a function $f \in C_0^\infty(\mathbb{R})$ there are many ways to construct an almost analytic extension $\widetilde{f} \in C_0^\infty(\mathbb{C})$ of f. A simple explicit way is given by the following proposition :

Proposition. Let $f \in C_0^\infty(\mathbb{R})$. Let $\chi, \psi \in C_0^\infty(\mathbb{R})$ such that

$$\chi \equiv 1 \text{ near } 0, \quad \psi \equiv 1 \text{ near the support of } f.$$

Then

$$\widetilde{f}(x+iy) = \frac{\psi(x)\chi(y)}{2\pi} \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi) \widehat{f}(\xi) d\xi$$

is an almost analytic extension of f. Here $\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-it\xi} f(t) dt$ stands for the Fourier transform of f.

Proof. The first property holds immediately by the Fourier inversion formula : we have for all $x \in \mathbb{R}$

$$\widetilde{f}(x+i0) = \frac{\psi(x)}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi = \psi(x) f(x) = f(x).$$

To prove the second property, notice that since $x+iy \mapsto e^{x+iy\xi}$ is holomorphic, $\overline{\partial}\widetilde{f}$ is, up to the constant $\frac{1}{4\pi}$, the sum of the following three terms

$$I := \psi'(x)\chi(y) \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi) \widehat{f}(\xi) d\xi$$

$$II := i\psi(x)\chi'(y) \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi) \widehat{f}(\xi) d\xi$$

$$III := i\psi(x)\chi(y) \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi'(y\xi) \xi \widehat{f}(\xi) d\xi$$

We have then to estimate each one of these terms.

Almost analytic extensions

• In the first term I, we expand $e^{-y\xi}\chi(y\xi)$ by the Taylor formula which gives a remainder of the form

$$\psi'(x)\chi(y)\int_{\mathbb{R}} e^{ix\xi}\mathcal{O}(|y\xi|^N)\widehat{f}(\xi)d\xi = \mathcal{O}(|y|^N)$$

and a linear combination of terms of the form

$$\psi'(x)\chi(y)\int_{\mathbb{R}} e^{ix\xi}(y\xi)^k\widehat{f}(\xi)d\xi = 0$$

since by the Fourier inversion formula, the integral equals $y^k f^{(k)}(x)$ up to a multiplicative constant and since ψ' vanishes on the support of f.

• Obviously II vanishes near $y = 0$.

• In the last term III, using that χ' vanishes near 0, the integral can be written as

$$\int_{\mathbb{R}} e^{(x+iy)\xi}\frac{\chi'(y\xi)}{(y\xi)^N}y^N\xi^{N+1}\widehat{f}(\xi)d\xi = \mathcal{O}(|y|^N).$$

Remark : From the above construction we notice the following :

- An almost analytic extension is not unique.
- We can take $\widetilde{f}$ with support in an arbitrarily small neighborhood of the support of f.
- If g is the difference of two almost analytic extensions of the same function, so that $g_{|\mathbb{R}} = 0$ and $\overline{\partial}g = \mathcal{O}(|\mathrm{Im}\, z|^N)$, or all $N \in \mathbb{N}$, then $g = \mathcal{O}(|\mathrm{Im}\, z|^N)$ for all $N \in \mathbb{N}$.
- One can actually construct almost analytic extensions for a more general class of functions than $C_0^\infty(\mathbb{R})$ (for instance Schwartz functions or even more general). But for our applications this class is sufficient.

Integrals involving almost analytic extensions

For a continuous function $B(x,y)$ defined on $\mathbb{R}^2\setminus\{y=0\}$, or equivalently on $\mathbb{C}\setminus\mathbb{R}$, with values in a Banach space, we shall denote

$$\int_{|\mathrm{Im}\, z|\geq\varepsilon}\overline{\partial}\widetilde{f}(z)B(z)L(dz):=\int_{|y|\geq\varepsilon}\left(\int_{\mathbb{R}}\overline{\partial}\widetilde{f}(x,y)B(x,y)dx\right)dy$$

and

$$\int_{\mathbb{C}}\overline{\partial}\widetilde{f}(z)B(z)L(dz):=\lim_{\varepsilon\to 0}\int_{|\mathrm{Im}\, z|\geq\varepsilon}\overline{\partial}\widetilde{f}(z)B(z)L(dz).$$

The following result justifies the existence of integrals involving almost analytic extensions :

Proposition. Let $f\in C_0^\infty(\mathbb{R})$ and $\widetilde{f}$ an almost analytic extension of f supported in $[a,b]+i[c,d]$. For all continuous function $B:[a,b]+i[c,d]\setminus\mathbb{R}\to\mathcal{B}$ with values in a Banach space $\mathcal{B}$ and such that, for some constants $C,M>0$,

$$\|B(z)\|_{\mathcal{B}}\leq C|\mathrm{Im}\, z|^{-M},\quad z\in[a,b]+i[c,d]\setminus\mathbb{R},$$

the following hold

- The integral
$$\int_{\mathbb{C}}\overline{\partial}\widetilde{f}(z)B(z)L(dz)$$
is well defined.
- We have the bound
$$\left\|\int_{\mathbb{C}}\overline{\partial}\widetilde{f}(z)B(z)L(dz)\right\|_{\mathcal{B}}\leq C\sup_{[a,b]+i[c,d]\setminus\mathbb{R}}\|(\mathrm{Im}\, z)^M B(z)\|_{\mathcal{B}}.$$

Proof. By standard results, the map $y\mapsto\int_{\mathbb{R}}\overline{\partial}\widetilde{f}(x,y)B(x+iy)dx$ is continuous on $[c,d]\setminus\{0\}$ and satisfies

$$\left\|\int_{\mathbb{R}}\overline{\partial}\widetilde{f}(x,y)B(x+iy)dx\right\|_{\mathcal{B}}\leq(b-a)\sup_{x\in[a,b]}\|y^M B(x+iy)\|_{\mathcal{B}}\sup_{x\in[a,b]}\|y^{-M}\overline{\partial}\widetilde{f}(x,y)\|_{\mathcal{B}}$$

for all $y\in[c,d]\setminus\{0\}$. The result follows easily after an integration w.r.t. y.

Helffer-Sjöstrand formula

Theorem. Let P be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $f \in C_0^\infty(\mathbb{R})$ and let $\widetilde{f} \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. Then

$$f(P) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(z-P)^{-1}L(dz). \tag{2}$$

Before proving this result, let us first give some important remarks.

- Notice that the left hand side of (2) is well defined by the spectral theorem. For the right hand side, we recall that $\|(z-P)^{-1}\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(|\mathrm{Im}\, z|^{-1})$ for $z \in \mathbb{C}\setminus\mathbb{R}$, hence the integral is well defined according to the previous proposition.

- The right hand side of (2) doesn't depends on the particular choice of the almost analytic extension $\widetilde{f}$: by Stoke's formula we have

$$I_{\widetilde{f}} := \lim_{\varepsilon\to 0} -\frac{1}{\pi}\int_{|\mathrm{Im}\, z|>\varepsilon} \overline{\partial}\widetilde{f}(z)(z-P)^{-1}L(dz) = \lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_{\mathbb{R}}\left[\widetilde{f}(z)(z-P)^{-1}\right]_{z=t-i\varepsilon}^{z=t+i\varepsilon} dt.$$

- For any real number λ and any integer $n \geq 0$, one has the following Cauchy type formula

$$\frac{(-1)^n}{n!}f^{(n)}(\lambda) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(z-\lambda)^{-1-n}L(dz).$$

Helffer-Sjöstrand formula

Proof of the Theorem. It suffices to apply the spectral theorem to both sides of the formula. More precisely, set

$$Q := -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(z-P)^{-1}L(dz) \in \mathcal{L}(\mathcal{H}).$$

Let $u, v \in \mathcal{H}$ and write

$$\langle (z-P)^{-1}u, v\rangle_{\mathcal{H}} = \int (z-t)^{-1}\langle dE_t u, v\rangle_{\mathcal{H}}$$

where $E_t = 1_{(-\infty,t]}(P)$ is the family of spectral projections associated with P. Consequently,

$$\langle Qu, v\rangle_{\mathcal{H}} = -\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)\int (z-t)^{-1}\langle dE_t u, v\rangle_{\mathcal{H}} L(dz)$$

and by Fubini's theorem one gets

$$\begin{aligned}\langle Qu, v\rangle_{\mathcal{H}} &= \int \left(-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(z-t)^{-1}L(dz)\right)\langle dE_t u, v\rangle_{\mathcal{H}} \\ &= \int f(t)\langle dE_t u, v\rangle_{\mathcal{H}} \\ &= \langle f(P)u, v\rangle_{\mathcal{H}}.\end{aligned}$$

This proves that $Q = f(P)$.

Using the Helffer-Sjöstrand formula we are now able to give an answer to the first question :

> Is $f(P)$ a semiclassical pseudodifferential operator if P is a semiclassical pseudodifferential operator and f is a "nice" function ?

Consider a self-adjoint semiclassical pseudodifferential operator $P = \mathrm{Op}_h^w(p)$ defined on a dense subspace of a Hilbert space $\mathcal{H}$. In order to simplify we shall always take $\mathcal{H} = L^2(\mathbb{R}^d)$. The symbol of the operator P is a smooth real-valued function $p : \mathbb{R}^{2d} \to \mathbb{R}$ which belongs to some class of symbols $S(m)$ for some order function $m \geq 1$:

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta} m(x,\xi), \quad \forall \alpha,\beta \in \mathbb{N}^d, x \in \mathbb{R}^d.$$

The typical example is the semiclassical Schrödinger operator

$$P = -h^2\Delta + V(x)$$

where $\Delta := \sum_{j=1}^d \partial_{x_j}^2$ is the Laplace operator in $\mathbb{R}^d$ and $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ is a smooth real-valued potential. The associated semiclassical symbol is given by $p(x,\xi) = \xi^2 + V(x)$, $(x,\xi) \in \mathbb{R}^{2d}$.

We can keep in mind as guiding examples :

- $V \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$.
- The harmonic oscillator $V(x) = \sum_{j=1}^d \mu_j x_j^2$, with $\mu_j > 0$, which is the natural approximation of a potential near its minimum (when non-degenerate).
- $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ of long-range type :

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\delta-|\alpha|}, \quad \forall x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

  for some $\delta > 0$.

The following result is the main result of the functional calculus of semiclassical pseudodifferential operators. This result is due to **Helffer-Robert** (1983).

Theorem. Let $f \in C_0^\infty(\mathbb{R})$. Then $f(P)$ is a semiclassical pseudodifferential operator with symbol in $S(m^{-k})$ for any $k \in \mathbb{N}$. More precisely, there exists a semiclassical symbol $a = a(x, \xi; h) \in \cap_{k\in\mathbb{N}} S(m^{-k})$ such that

$$f(P) = \mathrm{Op}_h^w(a)$$

Moreover the symbol a admits the following asymptotic expansion in powers of h

$$a(x, \xi; h) \sim \sum_{j\geq 0} h^j a_j(x, \xi) \quad \text{in } S(m^{-1}) \tag{3}$$

with $a_0(x, \xi) = f(p(x, \xi))$ and $a_1(x, \xi) = 0$.

Our objective now is to apply the functional calculus developed above to study the spectral properties of the operator P. Here we show an application for the study of the discrete spectrum.

Counting function of eigenvalues

Consider a self-adjoint semiclassical pseudodifferential operator $P(h) = \mathrm{Op}_h^w(p)$ in $L^2(\mathbb{R}^d)$ with real-valued symbol p. Let $[a, b] \subset \mathbb{R}$ be an interval and assume that for $h > 0$ small enough, the spectrum of $P(h)$ near $[a, b]$ is discrete :

$$\mathrm{Spec}\,(P(h)) \cap [a-\eta, b+\eta] = \{\lambda_1(h) \leq \lambda_2(h) \leq \cdots \leq \lambda_{N_h}(h)\}$$

for some $\eta > 0$, where each $\lambda_j(h)$ is a discrete eigenvalue of $P(h)$ of finite multiplicity.

As a consequence of the function calculus, this assumption is in general satisfied when $p^{-1}([a-\eta, b+\eta]) \subset \mathbb{R}^{2d}$ is compact. For instance :

- $P(h) = -h^2\Delta + V(x)$ with $V \in C^\infty(\mathbb{R}^{2d}; \mathbb{R})$ and $V(x) \to 0$ as $|x| \to +\infty$. The spectrum of $P(h)$ is discrete in any interval $[a, b]$ with $a < b < 0$.
- The harmonic oscillator $P(h) = -h^2\Delta + |x|^2$. The spectrum of P is discrete in $\mathbb{R}$.

The counting function of eigenvalues of $P(h)$ in $[a, b]$ is defined by

$$N_h(P, [a, b]) := \#\{j, \lambda_j(h) \in [a, b]\}.$$

This is just the number of eigenvalues of $P(h)$ in $[a, b]$ counting multiplicities. Notice that

$$N_h(P, [a, b]) = \mathrm{tr}\left[1_{[a,b]}(P(h))\right].$$

where $1_{[a,b]}$ is the characteristic function of $[a, b]$.

Theorem. Suppose that $p^{-1}([a-\eta, b+\eta]) \subset \mathbb{R}^{2d}$ is compact. Then for any $f \in C_0^\infty(]a,b[)$, the operator $f(P)$ is a trace class operator on $L^2(\mathbb{R}^d)$ and we have

$$\mathrm{tr}\,[f(P)] \sim (2\pi h)^{-d} \sum_{j=0}^{\infty} h^j \iint_{\mathbb{R}^{2d}} a_j(x,\xi)\,dx\,d\xi.$$

In particular

$$\mathrm{tr}\,[f(P)] = (2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} f(p(x,\xi))\,dx\,d\xi + \mathcal{O}(h^{-d+1}).$$

Proof. Let p_1 be a real-valued symbol such that $p_1(x,\xi) = p(x,\xi)$ for $|(x,\xi)|$ large enough and $\inf p_1 > b+\eta$. Then

1) P and $P_1 = \mathrm{Op}_h^w(p_1)$ are essentially self-adjoint in $L^2(\mathbb{R}^d)$ with the same domain.
2) $\exists$ an open neighborhood Ω of $[a-\eta, b+\eta]$ such that $(z-P_1)^{-1}$ is holomorphic for $z \in \Omega$.

Let $f \in C_0^\infty(]a,b[)$ and let $\tilde{f} \in C_0^\infty(\Omega)$ be an almost analytic extension of f such that $\mathrm{supp}\,\tilde{f} \subset \Omega$. We have

$$f(P_1) = 0$$

since $\mathrm{Spec}(P_1) \cap I = \emptyset$. For $\mathrm{Im}\, z \neq 0$, we have the resolvent identity

$$(z-P)^{-1} = (z-P_1)^{-1} + (z-P)^{-1}(P-P_1)(z-P_1)^{-1}.$$

Putting this into the Helffer-Sjöstrand formula we get

$$\begin{aligned} f(P) &= -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\left[(z-P_1)^{-1} + (z-P)^{-1}(P-P_1)(z-P_1)^{-1}\right] L(dz) \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\left[(z-P)^{-1}(P-P_1)(z-P_1)^{-1}\right] L(dz) \end{aligned}$$

Since $p-p_1$ is compactly supported, the operator $P-P_1$ is trace class with trace class norm $\mathcal{O}(h^{-d})$ so we conclude that $f(P)$ is of trace class and of trace class norm $\mathcal{O}(h^{-d})$.

To compute the trace, let $\chi \in C_0^\infty(\mathbb{R}^{2d})$ be equal to 1 in a neighborhood of $\mathrm{supp}\,(p-p_1)$. Then

$$f(P)(1-\mathrm{Op}_h^w(\chi)) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\left[(z-P)^{-1}(P-P_1)(z-P_1)^{-1}(1-\mathrm{Op}_h^w(\chi))\right] L(dz)$$

and

$$\|f(P)(1-\mathrm{Op}_h^w(\chi))\|_{\mathrm{tr}} = \mathcal{O}(h^N), \quad \forall N \in \mathbb{N}.$$

In particular for all $N \in \mathbb{N}$

$$\mathrm{tr}[f(P)] = \mathrm{tr}[f(P)\mathrm{Op}_h^w(\chi)] + \mathcal{O}(h^N)$$

We get the result from the previous results.

Weyl's law

Theorem. Under the above assumptions, one has as $h \to 0$

$$N_h(P,[a,b]) = \frac{1}{(2\pi h)^d}\left(\mathrm{Vol}\{(x,\xi)\in\mathbb{R}^{2d},\, p(x,\xi)\in[a,b]\} + o(1)\right)$$

Proof. Pick two sequences of compactly supported smooth functions $f_{1,\varepsilon}, f_{2,\varepsilon}$ that approaches the characteristic function $1_{[a,b]}$ from below and from above

$$1_{[a+\varepsilon,b-\varepsilon]} \leq f_{1,\varepsilon} \leq 1_{[a,b]} \leq f_{2,\varepsilon} \leq 1_{[a-\varepsilon,b+\varepsilon]}$$

Then we have

$$\mathrm{tr}\,[f_{1,\varepsilon}(P)] \leq \mathrm{tr}\left[1_{[a,b]}\right] = N_h(P(h),[a,b]) \leq \mathrm{tr}\,[f_{2,\varepsilon}(P)]$$

The conclusion follows from the trace formula of the above Theorem

$$\mathrm{tr}\,[f_{j,\varepsilon}(P)] = (2\pi h)^{-d}\iint_{\mathbb{R}^{2d}} f_{j,\varepsilon}(p(x,\xi))\,dx\,d\xi + \mathcal{O}(h^{-d+1})$$

by taking then limit $\varepsilon \to 0$.

Example : The harmonic oscillator

Consider the d-dimensional harmonic oscillator

$$P(h) = \frac{1}{2}\left(-h^2\Delta + |x|^2\right) = -\frac{h^2}{2}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2} + \frac{|x|^2}{2}, \quad x \in \mathbb{R}^d.$$

We want to compute all the eigenvalues of $P(h)$. To do so we introduce with the creation operators

$$C_k = \frac{1}{\sqrt{2}}\left(\frac{h}{i}\frac{\partial}{\partial x_k} + ix_k\right)$$

and the annihilation operators

$$A_k = C_k^* = \frac{1}{\sqrt{2}}\left(\frac{h}{i}\frac{\partial}{\partial x_k} - ix_k\right).$$

We have

$$[C_j, C_k] = 0 = [A_j, A_k] \quad \text{and} \quad [A_j, C_k] = h\delta_{j,k} \cdot \mathrm{Id}.$$

Moreover since

$$\begin{aligned} C_jA_j &= \frac{1}{2}\left(\frac{h}{i}\frac{\partial}{\partial x_j} + ix_j\right)\left(\frac{h}{i}\frac{\partial}{\partial x_j} - ix_j\right) \\ &= -\frac{h^2}{2}\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{2} - \frac{h}{2} \end{aligned}$$

we get

$$P(h) = \sum_{j=1}^{d} C_jA_j + \frac{d}{2}h \cdot \mathrm{Id} \tag{4}$$

and thus

$$[C_j, P(h)] = -hC_j \quad \text{and} \quad [A_j, P(h)] = hA_j.$$

It follows that if λ is an eigenvalue of P, that is $Pu = \lambda u$, then

$$PC_j u = [P, C_j]u + C_j Pu = hC_j u + C_j Pu = (\lambda + h)C_j u$$

and

$$PA_j u = (\lambda - h)A_j u.$$

This means that the creation operator C_j maps an eigenfunction u of P associated to the eigenvalue λ to an eigenfunction $C_j u$ of P associated to the eigenvalue $\lambda + h$, while the annihilation operator A_j maps u to an eigenfunction $A_j u$ (if it is nonzero) associated to the eigenvalue $\lambda - h$.

Observe from (4) that

$$\langle P(h)u, u\rangle = \sum_{j=1}^{d} \|A_j u\|^2 + \frac{d}{2}h\|u\|^2 \tag{5}$$

Then

1) If λ is an eigenvalue of P then $\lambda \geq \frac{d}{2}h$.
2) A function is an eigenfunction associated to $\lambda = \frac{d}{2}h$ if and only if $A_j u = 0$ holds for all $1 \leq j \leq n$.

A direct computation shows that

$$A_j u(x_j) = 0 \quad \Leftrightarrow \quad u(x_j) = ce^{-\frac{x_j^2}{2h}}.$$

and the we see that $\frac{d}{2}h$ is the smallest eigenvalue of P and the associated eigenfunction is $u_0(x) = e^{-\frac{|x|^2}{2h}}$ which is often called ground state.
Starting from the ground state, one can find more eigenvalues using the creation operator : for any $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, the function

$$u_\alpha = C_1^{\alpha_1} \cdots C_n^{\alpha_n} u_0$$

is an eigenfunction of P associated with the eigenvalue

$$\lambda_\alpha = \frac{d}{2}h + |\alpha|h.$$

Here $|\alpha| = \sum_{j=1}^{d} \alpha_j$.

The set $\{\lambda_\alpha = \frac{d}{2}h + |\alpha|h\}$ contains all the eigenvalues of $P(h)$.

Indeed, if $\lambda > \frac{d}{2}h$ is an eigenvalue of P with eigenfunction u, then by (5), there exists at least one $j \in \{1, ..., d\}$ such that $A_j u \neq 0$, and thus $\lambda - h$ is also an eigenvalue of P with eigenfunction $A_j u$. As a consequence

1) Any eigenvalue of P with $\lambda > \frac{d}{2}h$ must has the form $\lambda = \frac{d}{2}h + kh$ for some $k \in \mathbb{N}$.
2) If u is the corresponding eigenfunction, then there exists $\alpha_1, ..., \alpha_n \in \mathbb{N}$ with $|\alpha| = k$ such that $A^\alpha u := A_1^{\alpha_1} \cdots A_n^{\alpha_n} u = u_0$.

Finally, let us check Weyl's law for the harmonic oscillator. We have

$$\begin{aligned}\#\{j; \lambda_j \leq E\} &= \left\{\alpha; \frac{d}{2}h + |\alpha|h \leq E\right\} \\ &= \left\{\alpha; |\alpha| \leq \frac{1}{h}(E - \frac{d}{2}h)\right\} \\ &= \frac{1}{d!}\left(\frac{E}{h}\right)^d + o(h^{-d})\end{aligned}$$

while

$$\mathrm{Vol}\left\{(x,\xi) \in \mathbb{R}^{2d}, \frac{1}{2}(|\xi|^2 + |x|^2) \leq E\right\} = (\sqrt{2E})^{2d} \times \text{the volume of the unit ball in } \mathbb{R}^{2d}.$$

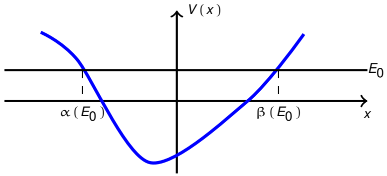
Weyl's law follows since the volume of the unit ball in $\mathbb{R}^{2d}$ is $\frac{\pi^d}{d!}$.

Consider the semiclassical self-adjoint one-dimensional Schrödinger operator in $L^2(\mathbb{R})$

$$P(h) := -h^2 \frac{d^2}{dx^2} + V(x)$$

with smooth real-valued potential V on the real line. Fix an energy-level $E_0 \in \mathbb{R}$ and suppose that $\liminf_{|x|\to+\infty} V(x) > E_0$ so that the spectrum of $P(h)$ consists of eigenvalues near E_0.

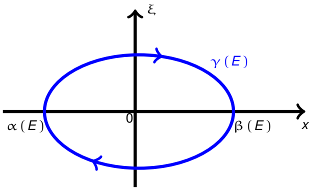
Simple well



For E close to E_0, the corresponding energy surface (the characteristic set)

$$\gamma(E) := \left\{(x,\xi) \in T^*\mathbb{R};\ p(x,\xi) = E\right\}, \quad p(x,\xi) := \xi^2 + V(x)$$

is a smooth simple closed curve in the phase space $T^*\mathbb{R} = \mathbb{R}^2_{x,\xi}$ symmetric w.r.t. the x-axis.



Here $\alpha(E)$ and $\beta(E)$ are the turning points, i.e. the zeros of $V(x) - E$ near $\alpha(E_0)$ and $\beta(E_0)$ respectively.

Action integral : The action along the curve $\gamma(E)$ is defined by the integral

$$\mathcal{A}(E) := \frac{1}{2}\int_{\gamma(E)} \xi dx = \int_{\alpha(E)}^{\beta(E)} \sqrt{E - V(x)}dx = \frac{1}{2}\int_{p(x,\xi)\le E} dxd\xi.$$

Bohr-Sommerfeld quantization rule. The eigenvalues of $P(h)$ near E_0 are approximated in the semiclassical limit $h \to 0^+$ by the roots of the equation

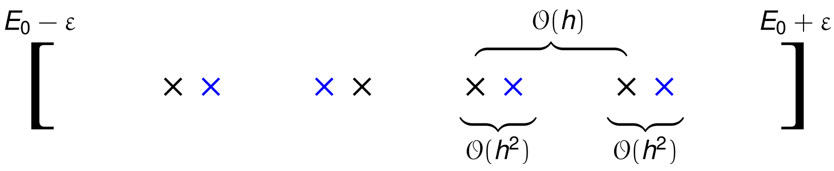
$$\cos\left(\frac{\mathcal{A}(E)}{h}\right) = 0. \tag{BS}$$

More precisely, we have

1) If $E = E(h)$ is an eigenvalue of $P(h)$ in a neighborhood of E_0, then there exists $k = k(E; h) \in \mathbb{N}$ such that
$$\mathcal{A}(E) = (k + \frac{1}{2})\pi h + \mathcal{O}(h^2).$$
2) Conversely, for every k such that $(k + \frac{1}{2})\pi h$ belongs to a neighborhood of $\mathcal{A}(E_0)$, there exists an eigenvalue $E_k(h)$ of $P(h)$ satisfying
$$\left|\mathcal{A}(E_k(h)) - (k + \frac{1}{2})\pi h\right| \leq Ch^2,$$
with a constant $C > 0$ independent of k and h.

Remarks :

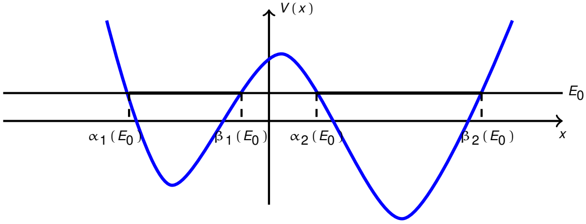
- There is exactly one eigenvalue of $P(h)$ in a neighb. of size $\mathcal{O}(h^2)$ of every root of (BS).
- The distance between two successive roots of (BS) is $\sim \frac{\pi}{\mathcal{A}'(E_0)} h$.

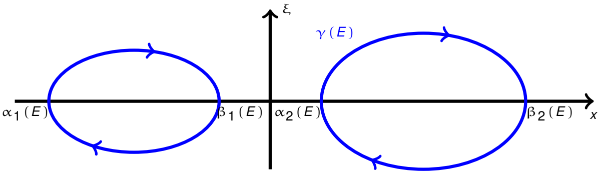


$\times$ = Roots of (BS) ; $\times$= Eigenvalues of $P(h)$

Example : **harmonic oscillator** $V(x) = x^2$. In this case $\mathcal{A}(E) = \frac{\pi E}{2}$ and $\sigma(P(h)) = \left\{(2k+1)h, k \in \mathbb{N}\right\}$.

Double well





The corresponding energy surface $\gamma(E) := \{(x,\xi) \in T^*\mathbb{R};\ \xi^2 + V(x) = E\}$ for E close to E_0.

Theorem (Helffer-Sjöstrand '84) The spectrum of $P(h)$ near E_0 is exponentially close, in the semiclassical limit $h \to 0^+$, to the union of the spectra of P restricted to each well :

$$\exists \text{ a bijection } \quad b_h : \sigma(P) \text{ near } E_0 \to \bigcup_{j=1,2} \sigma(P_{|(\alpha_j,\beta_j)}) \text{ near } E_0$$

$$b_h(E) - E = \mathcal{O}(e^{-S(E)/h}) \quad \text{as} \quad h \to 0^+,$$

where

$$S(E) := \int_{\beta_1(E)}^{\alpha_2(E)} \sqrt{V(x) - E}\,dx$$

is the so-called Agmon distance.

▶ (BS) quantization rule for a simple well : the eigenvalues of $P_{|(\alpha_j,\beta_j)}$ near E_0 are approximated in the semiclassical limit $h \to 0^+$ by

$$\mathcal{U}_h^{(j)} := \left\{ E \text{ near } E_0; \cos\left(\frac{\mathcal{A}_j(E)}{h}\right) = 0 \right\} \quad (j = 1, 2)$$

where $\mathcal{A}_j(E) := \int_{\alpha_j(E)}^{\beta_j(E)} \sqrt{E - V(x)}dx$ is the action associated with the well $(\alpha_j(E), \beta_j(E))$.

Assume that V is even (symmetric wells). In this case $\mathcal{U}_h^{(1)} = \mathcal{U}_h^{(2)}$.

▶ **LANDAU-LIFCHITZ '48 :** Quantum tunneling effect between the two wells appears as eigenvalue splitting.

Theorem (Gérard-Grigis '88) The eigenvalues of $P(h)$ near E_0 come out in pairs with splitting exponentially small. If $E_+(h)$ and $E_-(h)$ are a pair of eigenvalues which are exponentially close to $E \in \mathcal{U}_h^{(1)} \cup \mathcal{U}_h^{(2)}$, then

$$|E_+(h) - E_-(h)| = \frac{2h}{\mathcal{A}'(E)} e^{-S(E)/h} + \mathcal{O}(e^{-2S(E)/h}) \quad (h \to 0^+)$$

where $\mathcal{A}(E) := \mathcal{A}_1(E) = \mathcal{A}_2(E)$.

Remark : $\mathcal{A}'(E) = \frac{T(E)}{2}$ where $T(E)$ is the period of the Hamiltonian flow $(x(t), \xi(t))$ associated with the classical Hamiltonian $\xi^2 + V(x)$.